Lab 6 Notes

#1) $y'' = -by' - cy$. Letting $v = y'$, $\underbrace{\begin{bmatrix} y \\ v \end{bmatrix}'}_{X'} = \begin{bmatrix} v \\ -by' - cy \end{bmatrix} = \begin{bmatrix} v \\ -cy - bv \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -c & -b \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} y \\ v \end{bmatrix}}_{X}$

$p(\lambda) = \det(A - \lambda I) = (\text{show work}) = \lambda^2 + b\lambda + c$

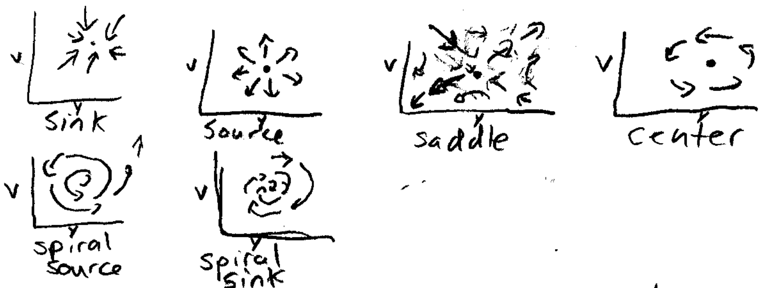
#2) $p(\lambda) \overset{\text{set}}{=} 0$ gives $\lambda = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$. If $b^2 - 4c < 0$ or $c > \frac{b^2}{4}$, λ is imaginary.

Note matlab syntax: "fplot('some_function', [a, b, c, d])" will plot the function for $a \le x \le b$, $c \le y \le d$.

...

#3) Types of equilibrium points:

See also pp. 379-380 in textbook. "Unstable" nodes usually refers to a source, and "stable" nodes are sinks.



#4.) If eigenvalues of A are λ_1 and λ_2, and corresponding eigenvectors
#5.) are $\vec{v}_1$ and $\vec{v}_2$, your general solution is given by
#6.)

$$\vec{X}(t) = c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2, \text{ if } \lambda_1 \neq \lambda_2 \quad (*)$$

or $$\vec{X}(t) = c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 t e^{\lambda_2 t} \vec{v}_2, \text{ if } \lambda_1 = \lambda_2$$

where c_1 and c_2 are arbitrary (cf. textbook pp. 367-368).

Note, if $\lim_{t \to \infty} \vec{X}(t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, then $\vec{X}(t) = \begin{bmatrix} y(t) \\ v(t) \end{bmatrix}$ is converging to the origin.

If any solution (the general solution) converges to $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, then the origin is a sink.

If $\lim_{t \to \infty} \vec{X}(t) = \begin{bmatrix} \infty \\ \infty \end{bmatrix}$, you can conclude the origin is a source.

#7.) Straight line solutions are instances of the general solutions where either $c_1 = 0$, $c_2 \neq 0$, or $c_1 \neq 0$, $c_2 = 0$ in $(*)$. Both of these are solutions, so if one straight line solution goes away from the origin, and the other straight line solution converges to $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, as $t \to \infty$, then the origin will be a source.

#8.) Notice that if $b^2 - 4c < 0$, the $\lambda = \frac{-b}{2} \pm \frac{\sqrt{b^2 - 4c}}{2} = -\frac{b}{2} \pm i \frac{\sqrt{|b^2 - 4c|}}{2} = \alpha \pm \beta i$,

where $\alpha = -\frac{b}{2}$ and is real, and $\beta = \frac{\sqrt{|b^2 - 4c|}}{2}$ and βi is imaginary.

Note that if $\lambda = \alpha \pm \beta i$, and $\lambda_1 = \alpha + \beta i$, $\lambda_2 = \alpha - \beta i$

then

$$\begin{bmatrix} y(t) \\ v(t) \end{bmatrix} = \vec{X}(t) = c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2 = c_1 e^{(\alpha + \beta i)t} \vec{v}_1 + c_2 e^{(\alpha - \beta i)t} \vec{v}_2$$

$$= e^{\alpha t} [c_1 e^{\beta i t} \vec{v}_1 + c_2 e^{-\beta i t} \vec{v}_2]$$

$$= e^{\alpha t} [d_1 \cos(\beta t) \vec{v}_1 + d_2 \sin(\beta t) \vec{v}_2]. \text{ where } d_1, d_2 \text{ are arbitrary constants.}$$

$\therefore$ $y(t) = e^{\alpha t} [m_1 \cos(\beta t) + m_2 \sin(\beta t)]$ for arbitrary constants m_1, m_2

$v(t) = e^{\alpha t} [n_1 \cos(\beta t) + n_2 \sin(\beta t)]$ for arbitrary constants n_1, n_2

These are useful formulas for analyzing the long-term behavior of y and v.